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The Limit Distribution of the Largest Interpoint Distance from a Symmetric Kotz Sample

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Generalizing recent work of P. C. Matthews and A. L. Rukhin (*Ann. Appl. Probab.* **3** (1993), 454–466), we obtain the limit law of the largest interpoint Euclidean distance for a spherically symmetric multivariate sample of the Kotz distribution. While going through the proof, some errors in the reasoning given by Matthews and Rukhin are pointed out and corrected. © 1996 Academic Press, Inc.

1. INTRODUCTION AND SUMMARY

Let $X_1, X_2, \ldots$ be a sequence of independent d-variate random vectors $(d \geqslant 2)$ having a spherical symmetric distribution around some point which without loss of generality may be taken to be the origin in $\mathbb{R}^d$. This paper is concerned with the limiting behavior of the largest interpoint distance

$$M_n = \max_{1 \leqslant i < j \leqslant n} |X_i - X_j|$$

as $n \to \infty$, where $|\cdot|$ is the Euclidean norm.

In the univariate case $d = 1$, the random variable M_n is the sample range $\max_{1 \leqslant j \leqslant n} X_j - \min_{1 \leqslant j \leqslant n} X_j$, and its asymptotic behavior is well known under general conditions on the underlying distribution of the X_j (see, e.g., Galambos [4]).

The multivariate case is much more difficult due to the fact that M_n is a maximum of random variables with a complicated dependence structure. Therefore, it is not surprising that very little is known about the behavior of M_n for $d \geqslant 2$, although this statistic has interesting applications in gun quality control (Matthews and Rukhin [6]) and outlier detection (Barnett and Lewis [2, Chap. 9.3]). In a fundamental paper Matthews and Rukhin

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[6] obtained the limiting distribution of M_n in the multivariate case under the special assumption of (spherically symmetric) normality.

The purpose of the present work is twofold. First, we generalize the result of Matthews and Rukhin by embedding the multivariate normal law into a parametric family of spherically symmetric distributions. In the second place, we point out (and correct) some errors in the reasoning given by Matthews and Rukhin while going through the proof.

To be more specific, we assume that the common distribution of the X_j is the symmetric Kotz type distribution $\mathscr{MK}_d(b, \kappa, 1, \mathbf{0}, I_d)$ with density

$$f(x) = \frac{\kappa^{d/2+b-1}\Gamma(d/2)}{\pi^{d/2}\Gamma(d/2+b-1)}\,|x|^{2(b-1)}\exp(-\kappa\,|x|^2), \qquad x \in \mathbb{R}^d,$$

where $d \geqslant 2$, $2b+d>2$, and $\kappa>0$ (see Fang, Kotz, and Ng [3, Section 3.2]). This class includes the d-variate standard normal distribution as a special case for $b=1$ and $\kappa=\frac{1}{2}$. It has been used as an alternative model to the normal distribution in the context of testing for multivariate normality (see, e.g., Baringhaus and Henze [1]).

Put

$$r_1^2 = r_1^2(n) = \kappa^{-1}[\log n + \tfrac{1}{4}(d+4b-7)\,l_2 n + \tfrac{1}{2}(l_3 n + a + c)], \qquad (1.1)$$

where c is a fixed real number,

$$a = a(d, b) = \log \frac{(d-1)\,2^{(d-7)/2}\,\Gamma(d/2)}{\pi^{1/2}\Gamma^2(d/2+b-1)} \qquad (1.2)$$

and, for short, $l_2 n = \log\log n$ and $l_3 n = \log l_2 n$. In what follows, we tacitly assume that natural logarithms are used and that n is sufficiently large. Denoting by

$$D_n = \operatorname{card}\{(i, j) : 1 \leqslant i < j \leqslant n,\ |X_i - X_j| \geqslant 2r_1\}$$

the number of exceedances by the interpoint distances of the level $2r_1$, our main result is as follows.

THEOREM 1. *As n tends to infinity, D_n converges in distribution to a Poisson random variable with parameter e^{-c}.*

Due to the equality $\{D_n > 0\} = \{M_n > 2r_1\}$, Theorem 1 has the following corollary.

COROLLARY 1. *We have*

$$\lim_{n\to\infty} P\left(M_n^2 \leqslant \frac{4}{\kappa}\left[\log n + \frac{1}{4}(d+4b-7)\, l_2 n + \frac{1}{2}(l_3 n + a + c)\right]\right)$$

$$= \lim_{n\to\infty} P\left(\sqrt{(1/\kappa)\log n}\left[M_n - 2\sqrt{(1/\kappa)\log n}\right.\right.$$

$$\left.\left. - \frac{(1/2)(d+4b-7)\, l_2 n + l_3 n + a}{\sqrt{4\kappa \log n}}\right] \leqslant \frac{c}{2\kappa}\right)$$

$$= \exp(-e^{-c}).$$

To prove Corollary 1, note that the second probability equals

$$P\left(M_n^2 \leqslant \frac{4}{\kappa}\left[\log n + \frac{1}{4}(d+4b-7)\, l_2 n + \frac{1}{2}(l_3 n + a + c)\right] + \varepsilon_n\right),$$

where

$$\varepsilon_n = (4\kappa \log n)^{-1}\left(\tfrac{1}{2}(d+4b-7)\, l_2 n + l_3 n + a + c\right)^2$$
$$= o(1).$$

Remark 1. Putting $\kappa = \frac{1}{2}$ and $b = 1$, Theorem 1 yields Theorem 1 of Matthews and Rukhin [6]. However, our Corollary 1 differs from their Corollary 1 in that they have a superfluous factor 2 in the denominator, figuring in the third line on page 456 of their paper (this factor erroneously also occurs in the corresponding expression for the univariate case given on page 454).

Remark 2. Before going into details, it is convenient to point out that the role of κ is merely that of a scale parameter. In fact, letting $\tilde{X}_j = \sqrt{2\kappa}\, X_j$, $j \geqslant 1$, and $\tilde{M}_n = \max_{1 \leqslant i < j \leqslant n} |\tilde{X}_i - \tilde{X}_j|$, we have $\tilde{X}_j \sim \mathscr{MK}_d(b, \frac{1}{2}, 1, \mathbf{0}, I_d)$ and $\tilde{M}_n = \sqrt{2\kappa}\, M_n$. Consequently, we put $\kappa = \frac{1}{2}$ for convenience in what follows. This choice of κ entails that $|X_j|^2$ has the gamma density:

$$f_r(y) = \left[2^{d/2+b-1}\Gamma(d/2+b-1)\right]^{-1} y^{d/2+b-2} e^{-y/2}, \qquad y > 0. \qquad (1.3)$$

Just like in Matthews and Rukhin [6], the fundamental idea for proving Theorem 1 is to split up the sum

$$D_n = \sum_{1 \leqslant i < j \leqslant n} \mathbf{1}\{|X_i - X_j| \geqslant 2r_1\}$$

of indicators into four terms, each of which corresponds to possible "sources of exceedances." Of these terms three are shown to be asymptotically negligible, and the dominating term may be tackled using a Poisson limit theorem for U-statistics from Silverman and Brown [7].

To be precise, we consider the four radii r_0, r_1, r_2, r_3 defined by

$$r_1^2 = r_1^2(n) = 2\log n + \tfrac{1}{2}(d+4b-7)\, l_2 n + l_3 n + a + c$$

(this is (1.1) with $\kappa = \frac{1}{2}$, cf. Remark 2),

$$r_2^2 = r_2^2(n) = 2\log n + \frac{1}{2}(d+4b-7)\, l_2 n + \frac{d}{2}\, l_3 n + 2(a+c),$$

$$r_3^2 = r_3^2(n) = 2\log n + (d+2b-4)\, l_2 n + 2 l_3 n$$

and

$$r_0 = r_0(n) = 2r_1 - r_2.$$

Since

$$(2\log n + O(l_2 n))^{1/2} = (2\log n)^{1/2} \cdot (1 + O(l_2 n/\log n)),$$

all these radii are in a narrow annulus at $(2\log n)^{1/2} + O(l_2 n \log n)^{-1/2})$. Possible exceedances are eliminated as follows. Letting

$$E(i, j) = \{ |X_i - X_j| \geqslant 2r_1 \}, \tag{1.4}$$

it follows that

$$D_n = A_n + B_n + C_n + \bar{D}_n, \tag{1.5}$$

where

$$A_n = \sum_{1 \leqslant i < j \leqslant n} \mathbf{1}\,[E(i, j) \cap (\{ |X_i| \geqslant r_3 \} \cup \{ |X_j| \geqslant r_3 \})],$$

$$B_n = \sum_{1 \leqslant i < j \leqslant n} \mathbf{1}\,[E(i, j) \cap \{ r_2 \leqslant |X_i| \leqslant r_3 \} \cap \{ r_2 \leqslant |X_j| \leqslant r_3 \}],$$

$$C_n = \sum_{1 \leqslant i < j \leqslant n} \mathbf{1}\,[E(i, j) \cap \{ |X_i| \leqslant r_2 \} \cap \{ |X_j| \leqslant r_2 \}],$$

$$\bar{D}_n = \sum_{1 \leqslant i < j \leqslant n} \mathbf{1}\,[E(i, j) \cap (\{ |X_i| \leqslant r_2 \leqslant |X_j| \leqslant r_3 \} \cup \{ |X_j| \leqslant r_2 \leqslant |X_i| \leqslant r_3 \})].$$

We will show that the first three terms on the right-hand side of (1.5) converge stochastically to zero and that the limit distribution of $\bar{D}_n$ is Poisson with parameter e^{-c}.

2. Proofs

Proposition 1. *We have* $A_n \xrightarrow{\mathrm{P}} 0$ *as* $n \to \infty$.

Proof. Fix $\varepsilon > 0$. Since $\{A_n \geqslant \varepsilon\} \subset \bigcup_{j=1}^{n} \{|X_j| \geqslant r_3\}$, it follows that

$$P(A_n \geqslant \varepsilon) \leqslant nP(|X_1| \geqslant r_3).$$

Recalling the density f_r of $|X_1|^2$ given in (1.3), an appeal to the standard formula

$$\int_x^{\infty} e^{-t} t^{\alpha-1}\, dt = x^{\alpha-1} e^{-x}(1 + O(x^{-1})), \qquad x \to \infty, \tag{2.1}$$

for the incomplete gamma function (see formula (8.357) of Gradshteyn and Ryzhik [5]) and straightforward algebra yield $nP(|X_1|^2 \geqslant r_3^2) = O((l_2 n)^{-1})$. ▮

Remark 3. Note that for the "normal case" considered by Matthews and Rukhin [6] we have $b = 1$ and, thus,

$$r_3^2 = 2 \log n + (d-2)\, l_2 n + 2 l_3 n$$

Instead of r_3 Matthews and Rukhin used the slithtly greater radius $\tilde{r}_3$ defined by

$$\tilde{r}_3^2 = 2 \log n + d l_2 n \tag{2.2}$$

and obtained the smaller bound $nP(|X_1|^2 \geqslant \tilde{r}_3^2) = O((\log n)^{-1})$.

Before going on with the proof, we state some auxiliary results.

Proposition 2. *Let* X *and* Y *be independent random vectors following the symmetric Kotz distribution* $\mathscr{MK}_d(b, \frac{1}{2}, 1, \mathbf{0}, I_d)$, *and let* $W = |X|^2 - r_2^2$, $Z = r_2^2 - |Y|^2$. *The conditional densities of* W *and* Z *are, as* $n \to \infty$,

$$f_1(w \mid r_2^2 \leqslant |X|^2 \leqslant r_3^2) = \tfrac{1}{2} e^{-w/2}[1 + o(1)], \qquad 0 < w < r_3^2 - r_2^2$$
$$f_2(z \mid r_0^2 \leqslant |Y|^2 \leqslant r_2^2) = \tfrac{1}{2} e^{-(r_2^2 - r_0^2 - z)/2}[1 + o(1)], \qquad 0 < z < r_2^2 - r_0^2.$$

Proof. Use (1.3) and (2.1) (see also Proposition 3 of Matthews and Rukhin [6]). ▮

Proposition 3. *Let* U *and* V *be independently uniformly distributed on the spheres of radii* $(r_2^2 + x)^{1/2}$ *and* $(r_2^2 + y)^{1/2}$, *respectively. If* x *and* y *are both of order* $O(l_2 n)$ *and*

$$(r_2^2 + x)^{1/2} + (r_2^2 + y)^{1/2} \geqslant 2 r_1,$$

then

$$P(|U-V|\geqslant 2r_1)=\frac{1}{K}\left(\frac{(d-2)\,l_3 n+2(a+c)+x+y+o(1)}{\log n}\right)^{(d-1)/2} \times (1+o(1)) \tag{2.3}$$

as $n\to\infty$, *where* $K=(d-1)\,B(1/2,(d-1)/2)$ *and* $B(\cdot,\cdot)$ *is the beta function.*

Proof. The proof is completely analogous to the proof of Proposition 4 of Matthews and Rukhin [6]. ∎

Remark 4. The main difference between (2.3) and formula (2.6) of Matthews and Rukhin [6] is that we have the coefficient $d-2$ (instead of 2) of $l_3 n$. This is due to the fact that Matthews and Rukhin use a radius $\tilde{r}_2$ (their r_2 given in (1.4)) defined by

$$\tilde{r}_2^2=2\log n+\tfrac{1}{2}(d-3)\,l_2 n+2(l_3 n+a+c), \tag{2.4}$$

whereas our choice of r_2 in the "normal case" $b=1$ gives

$$r_2^2=2\log n+\frac{1}{2}(d-3)\,l_2 n+\frac{d}{2}\,l_3 n+2(a+c) \tag{2.5}$$

which is slightly different from (2.4) if $d\neq 4$. There is a misprint in formula (1.4) of Matthews and Rukhin in that the second $l_2 n$ must be replaced by $l_3 n$. This becomes clear from the expression for t_1 given on page 459 of their paper.

PROPOSITION 4. *For* B_n *defined in* (1.5) *we have*

$$B_n \xrightarrow{\mathrm{P}} 0 \qquad as \quad n\to\infty.$$

Proof. Note that, for $0<\varepsilon\leqslant 1$,

$$\begin{aligned} P(B_n\geqslant\varepsilon)&\leqslant E(B_n)\\ &=\binom{n}{2}P(|X_1-X_2|\geqslant 2r_1,\, r_2\leqslant|X_1|,\,|X_2|\leqslant r_3)\\ &=\binom{n}{2}\cdot p^2\cdot P(|X_1-X_2|\geqslant 2r_1\mid r_2\leqslant|X_1|,\,|X_2|\leqslant r_3), \end{aligned}$$

where

$$p=P(r_2\leqslant|X_1|\leqslant r_3).$$

Since $p^{-1}f_r(\xi)$, $r_2^2 \leqslant \xi \leqslant r_3^2$, is the conditional density of $|X_1|^2 - r_2^2$ given that $r_2 \leqslant |X_1| \leqslant r_3$ (cf. Proposition 2), a change of variable yields

$$E(B_n) = \binom{n}{2} \int_0^{r_3^2 - r_2^2} \int_0^{r_3^2 - r_2^2} f_r(r_2^2 + x)\, f_r(r_2^2 + y)$$
$$\times P(|X_1 - X_2| \geqslant 2r_1 \mid |X_1|^2 = r_2^2 + x,\ |X_2|^2 = r_2^2 + y)\, dy\, dx,$$

and, using (1.3) and Proposition 3, we obtain

$$E(B_n) = M(1 + o(1))n^2 \int_0^{r_3^2 - r_2^2} \int_0^{r_3^2 - r_2^2} [(r_2^2 + x)(r_2^2 + y)]^{d/2 + b - 2}$$
$$\times e^{-(2r_2^2 + x + y)/2} \left(\frac{(d-2)\, l_3 n + 2(a+c) + x + y + o(1)}{\log n} \right)^{(d-1)/2} dy\, dx,$$

where $M = [K 2^{d + 2b - 1} \Gamma^2(d/2 + b - 1)]^{-1}$ and K is given in Proposition 3. Finally, observing that $x = O(l_2 n)$, $y = O(l_2 n)$,

$$[(r_2^2 + x)(r_2^2 + y)]^{d/2 + b - 2} = O((\log n)^{d + 2b - 4})$$

and

$$\exp(-r_2^2) = O[n^{-2} \log n)^{-d/2 - 2b + 7/2} (l_2 n)^{-d/2}],$$

straightforward algebra gives

$$E(B_n) = O((l_2 n)^{-1/2}). \quad \blacksquare$$

Remark 5. There is an error in the stated order in Proposition 5 of Matthews and Rukhin [6]. In fact, bounding the double integral figuring on page 460 from below by replacing the lower integration bounds by $1/2(\tilde{r}_3^2 - \tilde{r}_2^2)$ (recall that $\tilde{r}_j$ is their r_j), we see that the exact order of $E(B_n)$ is $(l_2 n)^{(d-5)/2}$. The important consequence of this result is that B_n with r_j replaced by $\tilde{r}_j$ $(j = 2, 3)$ is asymptotically negligible if and only if $d \leqslant 4$. As a remedy we allow the coefficient of $l_3 n$ in (2.5) to depend on d.

PROPOSITION 5. *For C_n defined in* (1.5) *we have*

$$C_n \xrightarrow{\text{P}} 0 \qquad as \quad n \to \infty.$$

Proof. Proceeding completely analogously to the reasoning given in Proposition 6 of Matthews and Rukhin [6] it follows that

$$E(C_n) = O(l_3 n / l_2 n)$$

as $n \to \infty$ which proves the assertion. $\blacksquare$

We now consider the interesting nonnegligible constituent part $\bar{D}_n$ of D_n defined in (1.5).

PROPOSITION 6. *As* $n \to \infty$, $\bar{D}_n$ *converges in distribution to a Poisson random variable with parameter* e^{-c}.

Proof. Note that $\bar{D}_n$ is a U-statistic with the symmetric kernel

$$g_n(x, y) = \mathbf{1}\,[\{|x-y| \geqslant 2r_1\} \cap (\{|x| \leqslant r_2 \leqslant |y| \leqslant r_3\} \cup \{|y| \leqslant r_2 \leqslant |x| \leqslant r_3\})],$$

depending on n. By Theorem A of Silverman and Brown [7] it thus suffices to show that

$$\lim_{n \to \infty} \binom{n}{2} E[g_n(X_1, X_2)] = e^{-c} \tag{2.6}$$

and

$$\lim_{n \to \infty} n^3 E[g_n(X_1, X_2)\, g_n(X_1, X_3)] = 0. \tag{2.7}$$

The proof of (2.6) runs completely along the lines of the reasoning of Matthews and Rukhin [6] (see the verification of their formula (3.1)) and will thus only be sketched. Note that

$$\begin{aligned} &E[g_n(X_1, X_2)] \\ &\quad = 2P(r_2 \leqslant |X_1| \leqslant r_3)\, P(|X_1 - X_2| \geqslant 2r_1, |X_2| \leqslant r_2 \mid r_2 \leqslant |X_1| \leqslant r_3), \end{aligned}$$

where

$$P(r_2 \leqslant |X_1| \leqslant r_3) = \frac{e^{-a-c}(\log n)^{(d-1)/4}}{\Gamma(d/2+b-1)\, n(l_2 n)^{d/4}}\,(1+o(1)) \tag{2.8}$$

and

$$\begin{aligned} &P(|X_1 - X_2| \geqslant 2r_1, |X_2| \leqslant r_2 \mid r_2 \leqslant |X_1| \leqslant r_3) \\ &\qquad = \int_0^{r_3^2 - r_2^2} \gamma(x)\, f_1(x \mid r_2 \leqslant |X_1| \leqslant r_3)\, dx \end{aligned} \tag{2.9}$$

with

$$\gamma(x) = \int_0^s P(|X_1 - X_2| \geqslant 2r_1, |X_2|^2 = r_2^2 - y \mid |X_1|^2 = r_2^2 + x)\, f_r(r_2^2 - y)\, dy \tag{2.10}$$

and $s=(d-2)\,l_3 n+2(a+c)+x+o(1)$. Using Proposition 3 and making the change of variable $u=(d-2)\,l_3 n+2(a+c)+x-y$ we obtain

$$\gamma(x)=\frac{2^{(d-3)/2}\Gamma(d/2)\,e^{x/2}(l_2 n)^{d/4-1}}{\pi^{1/2}\Gamma(d/2+b-1)\,n(\log n)^{(d-1)/4}}\,(1+o(1)), \tag{2.11}$$

which, invoking Proposition 2, shows that (2.9) equals

$$\frac{(d-1)\,2^{(d-7)/2}\,\Gamma(d/2)(l_2 n)^{d/4}}{\pi^{1/2}\Gamma(d/2+b-1)\,n(\log n)^{(d-1)/4}}\,(1+o(1)). \tag{2.12}$$

Multiplying these formulas and recalling (1.2), one obtains (2.6).

To show (2.7), note that

$$\begin{aligned}
&E[\,g_n(X_1, X_2)\,g_n(X_1, X_3)]\\
&\quad=P(E(1,2)\cap E(1,3)\cap B(2)\cap B(3)\cap C(1))\\
&\qquad+P(r_2\leqslant |X_1|\leqslant r_3)\\
&\qquad\times P(E(1,2)\cap E(1,3)\cap C(2)\cap C(3)\mid r_2\leqslant |X_1|\leqslant r_3)\\
&\quad=p_1+P(r_2\leqslant |X_1|\leqslant r_3)\,p_2,
\end{aligned}$$

say, where, in addition to the notation introduced in (1.4), $B(j)=\{r_2\leqslant |X_j|\leqslant r_3\}$ and $C(j)=\{|X_j|\leqslant r_2\}$. We bound p_1 by exploiting the crucial fact that

$$\begin{aligned}
&E(1,2)\cap E(1,3)\cap B(2)\cap B(3)\cap C(1)\\
&\qquad\subset E(1,2)\cap \overline{E(1,3)}\cap B(2)\cap B(3)\cap C(1)
\end{aligned} \tag{2.13}$$

where

$$\overline{E(1,3)}=\left\{\cos\phi\geqslant 1-\frac{(r_2+r_3)^2-4r_1^2}{2r_2(2r_1-r_3)}\right\}$$

and ϕ is the angle between X_1 and $-X_3$. To see this, observe that the event $E(1,3)$ is equivalent to

$$\left\{\cos\phi\geqslant 1-\frac{(|X_1|+|X_3|)^2-4r_1^2}{2\,|X_1|\,|X_3|}\right\}.$$

On bounding the last fraction from above by using the inequalities $r_2\leqslant |X_3|\leqslant r_3$, $|X_1|\leqslant r_2$, and

$$2r_1-r_3\leqslant |X_1-X_3|-r_3\leqslant |X_1|+|X_3|-r_3\leqslant |X_1|,$$

(2.13) follows. The independence of $\overline{E(1,3)}$ from the other events now gives the estimate

$$\begin{aligned} p_1 &\leqslant P(E(1,2) \cap \overline{E(1,3)} \cap B(2) \cap B(3) \cap C(1)) \\ &= P(\overline{E(1,3)}) \cdot P(B(2) \cap B(3) \mid \overline{E(1,3)}) \\ &\quad \times P(E(1,2) \cap C(1) \mid \overline{E(1,3)} \cap B(2) \cap B(3)) \\ &= P(\overline{E(1,3)})\, P(B(2) \cap B(3))\, P(E(1,2) \cap C(1) \mid B(2)). \end{aligned}$$

Since $\cos^2 \phi$ has the beta distribution $B(1/2, (d-1)/2)$ (see, e.g., Proposition 2 of Matthews and Rukhin [6]) and

$$\frac{(r_2+r_3)^2 - 4r_1^2}{2r_2(2r_1 - r_3)} = O\left(\frac{l_2 n}{\log n}\right),$$

a simple calculation shows that

$$P(\overline{E(1,3)}) = O\left(\left(\frac{l_2 n}{\log n}\right)^{(d-1)/2}\right).$$

Invoking (2.8) we have

$$P(B(2) \cap B(3)) = O(n^{-2} (\log n)^{(d-1)/2}\, (l_2 n)^{-d/2}),$$

and using (2.9), (2.12) it follows that

$$P(E(1,2) \cap C(1) \mid B(2)) = O(n^{-1} (l_2 n)^{d/4}\, (\log n)^{-(d-1)/4}).$$

On combining these estimates we obtain

$$p_1 = O(n^{-3} (l_2 n)^{(d-2)/4}\, (\log n)^{-(d-1)/4}). \tag{2.14}$$

To bound the conditional probability p_2, note that conditioning on $|X_1|$ yields

$$p_2 = \int_0^{r_3^2 - r_2^2} \gamma^2(x)\, f_1(x \mid r_2^2 \leqslant |X_1|^2 \leqslant r_3^2)\, dx$$

with $\gamma(x)$ given in (2.10). Using Proposition 2 and (2.11) it follows that

$$\begin{aligned} p_2 &= \frac{2^{d-3} \Gamma^2(d/2) (l_2 n)^{d/2-2}}{\pi \Gamma^2(d/2+b-1)\, n^2 (\log n)^{(d-1)/2}} \int_0^{r_3^2 - r_2^2} \frac{1}{2} e^{x/2}\, dx (1+o(1)) \\ &= O(n^{-2} (\log n)^{-(d-1)/4}\, (l_2 n)^{d/4-1}). \end{aligned}$$

Together with (2.8) we obtain

$$P(r_2 \leqslant |X_1| \leqslant r_3) \cdot p_2 = O(n^{-3}(l_2 n)^{-1}),$$

which, combined with (2.14), implies (2.7). ▮

Remark 6. There is a crucial error in formula (3.6) of Matthews and Rukhin [6]. With their definition of the radii r_2 and r_3 (see (2.2) and (2.4) of the present paper) the correct order in (3.6) is $O(n^{-2}(\log n)^{(d-5)/4}\,(l_2 n)^{-1})$ which is too large to make substantial parts of the proof go through. The remedy here is to make the radius r_3 a little smaller by subtracting $2l_2 n$ and adding the term $2l_3 n$ (see the definition of r_3^2 and $\tilde{r}_3^2$ in Remark 3).

3. Concluding Remarks

Remark 7. It is instructive to see how the parameter b which influence the tails of the underlying distribution enters into the limit behavior of M_n. Rewriting Corollary 1 with $\kappa = \frac{1}{2}$, we have

$$P\left(M_n \leqslant 2\sqrt{2\log n} + \frac{(1/2)(d+4b-7)\,l_2 n + l_3 n + a}{\sqrt{2\log n}} + \frac{c}{\sqrt{2\log n}}\right) \to e^{-e^{-c}}$$

which shows that varying b affects the coefficient of $l_2 n/\sqrt{2\log n}$ and the coefficient a of $(2\log n)^{-1/2}$ (see (1.2)). However, we have

$$M_n - 2\sqrt{2\log n} \xrightarrow{\mathrm{P}} 0$$

irrespective of b.

Remark 8. The proof of Theorem 1 shows that for any k with probability tending to 1 as $n \to \infty$, the k largest interpoint distances involve $2k$ distinct points.

As a consequence, an informal outlier test for (spherical) multivariate normality based on this observation (see the final section of Matthews and Rukhin [6]) is not able to detect nonnormal symmetric Kotz-type distributions. However, a clumping effect, i.e., the occurrence of a point with an exceptionally large norm leading to several of the largest interpoint distances, may be expected for heavytailed spherically symmetric distributions.

Remark 9. Simulation results indicate that the rate of convergence in Corollary 1 is at best logarithmic.

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References

[1] Baringhaus, L., and Henze, N. (1992). Limit distributions for Mardia's measure of multivariate skewness. *Ann. Statist.* **20** 1889–1902.

[2] Barnett, V., and Lewis, T. (1984). *Outliers in Statistical Data*, 2nd ed. Wiley, New York.

[3] Fang, K. T., Kotz, S., and Ng, K. W. (1989). *Symmetric Multivariate and Related Distributions*. Chapman & Hall, London.

[4] Galambos, J. (1978). *The Asymptotic Theory of Extreme Order Statistics*. Wiley, New York.

[5] Gradshteyn, I. S., and Ryzhik, I. M. (1980). *Table of Integrals, Series and Products*. Academic Press, New York.

[6] Matthews, P. C., and Rukhin, A. L. (1993). Asymptotic distribution of the normal sample range. *Ann. Appl. Probab.* **3** 454–466.

[7] Silverman, B., and Brown, T. (1978). Short distances, flat triangles, and Poisson limits. *J. Appl. Probab.* **15** 815–825.